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เอกลักษณ์ของฟังก์ชัน k -ฟีโบตรีโกณมิติ

Identities of k -Fibonometric Functions

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บทคัดย่อ

จุดประสงค์ของงานวิจัยนี้ได้กำหนดนิยาม k -ฟีโบนักซีไซน์ โคไซน์ แทนเจนต์ โคแทนเจนต์ และ เอกลักษณ์เบื้องต้นของฟังก์ชัน k -ฟีโบตรีโกณมิติ

คำสำคัญ : k -ฟีโบนักซี ฟังก์ชัน ; k -ฟีโบตรีโกณมิติ

Abstract

In this paper, the k -Fibonacci sine, cosine, tangent and cotangent are defined, and some identities of k -Fibonometric functions are investigated.

Keywords : k -Fibonacci numbers ; k -Fibonometric functions.

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Introduction

The well-known Fibonacci sequence is defined as $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ where $F_0 = 0$ and $F_1 = 1$. In a similar way, the k- Fibonacci sequence is defined as $F_{k,n} = kF_{k,n-1} + F_{k,n-2}$ for $n \geq 2$, $k \geq 1$ where $F_{k,0} = 0$ and $F_{k,1} = 1$. Its Binet's formula is given by $F_{k,n} = \dfrac{r_1^n - r_2^n}{r_1 - r_2}$ where $r_1 = \dfrac{k + \sqrt{k^2 + 4}}{2}$ and $r_2 = \dfrac{k - \sqrt{k^2 + 4}}{2}$ are the roots of the characteristic equation $r^2 - kr - 1 = 0$.

In 2001, Smith (2001) studied the Fibonometric function by investigations the initial value problem, $y'' - y' - y = 0$ with $y(0) = 0$ and $y'(0) = 1$, which is analogue to the definition of Fibonacci numbers $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ where $F_0 = 0$ and $F_1 = 1$. He defined the Fibonacci sine, cosine, tangent and cotangent and established some theorems and elementary identities for Fibonometry.

In this paper, the concept of k-Fibonometry is studied. The k-Fibonometric functions are obtained from a second order linear differential equation $y'' - ky' - y = 0$ which we call as k-Fibonometric differential equation. The initial value problem $y'' - ky' - y = 0$, $y(0) = 0$ and $y'(0) = 1$, known as k-Fibonometric differential equation, is analogous to the well known recursion formula for k-Fibonacci numbers $F_{k,n} = kF_{k,n-1} + F_{k,n-2}$ for $n \geq 2$, $k \geq 1$ where $F_{k,0} = 0$ and $F_{k,1} = 1$. In this paper, these functions will be defined as the k-Fibonometric functions.

Methods

The solution of k-Fibonometric differential equation is $y = \dfrac{e^{r_1 x} - e^{r_2 x}}{r_1 - r_2}$ where $r_1 = \dfrac{k + \sqrt{k^2 + 4}}{2}$ and $r_2 = \dfrac{k - \sqrt{k^2 + 4}}{2}$. Note that r_1 and r_2 are indeed the solutions of the equation $r^2 - kr - 1 = 0$. Motivated by the formula the well known $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$, we define the k- Fibonacci sine as follows:

Definition 2.1 The k - Fibonacci sine, denoted by $sinf_k$; is defined as $sinf_k(x) = \dfrac{e^{r_1 x} - e^{r_2 x}}{r_1 - r_2}$ where $r_1 = \dfrac{k + \sqrt{k^2 + 4}}{2}$ and $r_2 = \dfrac{k - \sqrt{k^2 + 4}}{2}$.

Before proving a theorem about k - Fibonacci sine, we have to prove the following lemmas.

Lemma 2.2 $(n + 2)(n + 1)c_{n+2} - k(n + 1)c_{n+1} - c_n = 0$ for all $n \geq 0$.

Proof: Suppose the solution of the k -Fibonometric differential equation $y'' - ky' - y = 0$ with $y(0) = 0$ and $y'(0) = 1$ is $y = \sum_{k=0}^{\infty} c_n x^n$.

Thus,

$$y' = \sum_{n=1}^{\infty} nc_n x^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}.$$

Hence

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - k\sum_{n=1}^{\infty} nc_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n - k\sum_{n=0}^{\infty} (n+1)c_{n+1} x^n - \sum_{n=0}^{\infty} c_n x^n = 0$$

so,

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)c_{n+2} - k(n+1)c_{n+1} - c_n\right]x^n = 0.$$

Therefore, $(n + 2)(n + 1)c_{n+2} - k(n + 1)c_{n+1} - c_n = 0$.

Lemma 2.3 $c_n = \dfrac{F_{k,n}c_1 + F_{k,n-1}c_0}{n!}, n \geq 1$ where $F_{k,n}$ is the n^{th} k -Fibonacci number.

Proof: We will prove by mathematical induction on n .

If $n = 1$ then $\dfrac{F_{k,1}c_1 + F_{k,0}c_0}{n!} = (1)c_1 + 0c_0 = c_1$.

Suppose that the hypothesis is true for $n = 1, 2, 3, \cdots, r, r + 1$.

Namely, $c_r = \dfrac{F_{k,r}c_1 + F_{k,r-1}c_0}{r!}$ and $c_{r+1} = \dfrac{F_{k,r+1}c_1 + F_{k,r}c_0}{(r+1)!}$.

We will show that it is true for $n = r + 2$.

From Lemma 2.2 $(r+2)(r+1)c_{r+2} - k(r+1)c_{r+1} - c_r = 0$.

Thus,

$$\begin{aligned}(r+2)(r+1)c_{r+2} &= k(r+1)c_{r+1} + c_r \\ &= k(r+1)\left(\frac{F_{k,r+1}c_1 + F_{k,r}c_0}{(r+1)!}\right) + \left(\frac{F_{k,r}c_1 + F_{k,r-1}c_0}{r!}\right) \\ &= \frac{kF_{k,r+1}c_1 + kF_{k,r}c_0 + F_{k,r}c_1 + F_{k,r-1}c_0}{r!}.\end{aligned}$$

Hence,

$$\begin{aligned}c_{n+2} &= \frac{\left(kF_{k,r+1} + F_{k,r}\right)c_1 + \left(kF_{k,r} + F_{k,r-1}\right)c_0}{(r+2)!} \\ &= \frac{F_{k,r+2}c_1 + F_{k,r+1}c_0}{(r+2)!}.\end{aligned}$$

Therefore, $c_n = \dfrac{F_{k,n}c_1 + F_{k,n-1}c_0}{n!},\ n \geq 1$.

Theorem 2.4 The expansion of the k -Fibonacci sine is $sinf_k(x) = \sum_{n=0}^{\infty} \dfrac{F_{k,n}}{n!}x^n$ where $F_{k,n}$ is the n^{th} k -Fibonacci number.

Proof: Since $y = \dfrac{e^{r_1x} - e^{r_2x}}{r_1 - r_2} = \sum_{k=0}^{\infty} c_n x^n$, we have

$$y = c_0 + c_1x + \left(\frac{F_{k,2}c_1 + F_{k,1}c_0}{2!}\right)x^2 + \cdots + \left(\frac{F_{k,n}c_1 + F_{k,n-1}c_0}{n!}\right)x^n + \cdots.$$

Imposing the initial conditions $y(0) = 0$ and $y'(0) = 1$ on the series, we have $c_0 = 0$ and $c_1 = 1$.

So,

$$\begin{aligned}y &= (1)x + \left(\frac{F_{k,2}}{2!}\right)x^2 + \left(\frac{F_{k,3}}{3!}\right)x^3 + \cdots + \left(\frac{F_{k,n}}{n!}\right)x^n + \cdots \\ &= \sum_{n=0}^{\infty} \frac{F_{k,n}}{n!}x^n.\end{aligned}$$

Therefore, $sinf_k(x) = \sum_{n=0}^{\infty} \frac{F_{k,n}}{n!} x^n$.

Next, we are moving forward to define the k -Fibonacci cosine, tangent and cotangent. We need to show that $sinf_k(x)$ is absolutely convergent for all real numbers x .

Lemma 2.5 $sinf_k(x)$ is absolutely convergent for all real numbers x .

Proof: We prove by using the ratio test.

$$\lim_{n\to\infty}\left|\frac{\frac{F_{k,n+1}}{(n+1)!}x^{n+1}}{\frac{F_{k,n}}{n!}x^{n}}\right| = \lim_{n\to\infty}\left|\frac{F_{k,n+1}x^{n+1}}{(n+1)!}\cdot\frac{n!}{F_{k,n}x^{n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x}{n+1}\cdot\frac{F_{k,n+1}}{F_{k,n}}\right|$$

$$= \lim_{n\to\infty}\frac{|x|}{n+1}\cdot\lim_{n\to\infty}\left(\frac{F_{k,n+1}}{F_{k,n}}\right)$$

$$= 0\cdot\lim_{n\to\infty}\left(\frac{F_{k,n+1}}{F_{k,n}}\right)$$

$$= 0\cdot\lim_{n\to\infty}\left[\left(\frac{r_1^{n+1}-r_2^{n+1}}{r_1-r_2}\right)\left(\frac{r_1-r_2}{r_1^{n}-r_2^{n}}\right)\right]$$

$$= 0\cdot\lim_{n\to\infty}\left(\frac{r_1^{n+1}-r_2^{n+1}}{r_1^{n}-r_2^{n}}\right) = 0\cdot\lim_{n\to\infty} r_1 = 0, \text{ since } |r_2| < 1.$$

Therefore, $sinf_k$ is absolutely convergent for all real numbers x.

We define $\frac{d}{dx}\sin f_k(x) = \cos f_k(x)$.

Definition 2.6 The k -Fibonacci cosine, denoted by $cosf_k$; is defined as $cosf_k(x) = \frac{r_1 e^{r_1 x} - r_2 e^{r_2 x}}{r_1 - r_2}$

where $r_1 = \frac{k+\sqrt{k^2+4}}{2}$ and $r_2 = \frac{k-\sqrt{k^2+4}}{2}$.

Theorem 2.7 The expansion of the k -Fibonacci cosine is $cosf_k(x) = \sum_{n=0}^{\infty} \frac{F_{k,n+1}}{n!} x^n$.

Proof: Since $sinf_k(x) = \sum_{k=0}^{\infty} \frac{F_{k,n}}{n!} x^n$ is absolutely convergent for all real numbers x, we have

$$\begin{aligned}
\frac{d}{dx} sinf_k(x) &= \frac{d}{dx}\Big(\sum_{n=0}^{\infty} \frac{F_{k,n}}{n!} x^n\Big) \\
&= \sum_{n=0}^{\infty} \frac{d}{dx} \frac{F_{k,n}}{n!} x^n \\
&= \sum_{n=0}^{\infty} n \frac{F_{k,n}}{n!} x^{n-1} \\
&= 0 + F_{k,1} + \frac{2F_{k,2}x}{2!} + \frac{3F_{k,3}x}{3!} + \cdots \frac{nF_{k,n}x}{n!} + \cdots \\
&= F_{k,1} + \frac{F_{k,2}}{1} x + \frac{F_{k,3}}{2!} x^2 + \frac{F_{k,4}}{3!} x^3 + \cdots + \frac{F_{k,n}}{(n-1)!} x^{n-1} + \cdots \\
&= \sum_{n=0}^{\infty} \frac{F_{k,n+1}}{n!} x^n .
\end{aligned}$$

Therefore, $cosf_k(x) = \sum_{k=0}^{\infty} \frac{F_{k,n+1}}{n!} x^n$.

We are going to define the k -Fibonacci tangent but we have to prove that $cosf_k(x) \neq 0$ for all real numbers x .

Lemma 2.8 $cosf_k(x) \neq 0$ for all real numbers x .

Proof: Suppose that $cosf_k(x) = 0$ for some x. Then, we have $cosf_k(x) = \frac{r_1 e^{r_1 x} - r_2 e^{r_2 x}}{r_1 - r_2} = 0$.

Thus, $r_1 e^{r_1 x} - r_2 e^{r_2 x} = 0$ so, $\frac{r_2}{r_1} = e^{(r_1 - r_2)x}$.

Since

$$\frac{r_2}{r_1} = \frac{k - \sqrt{k^2+4}}{k + \sqrt{k^2+4}} = \frac{\left(k - \sqrt{k^2+4}\right)^2}{k^2 - (k^2+4)} = \frac{\left(k - \sqrt{k^2+4}\right)^2}{-4} = -(r_2)^2 = -r_2^2 < 0$$

but $\frac{r_2}{r_1} = e^{(r_1 - r_2)x} > 0$, we have a contradiction.

Therefore, $cosf_k(x) \neq 0$ for all real numbers x .

Definition 2.9 The k -Fibonacci tangent, denoted by $tanf_k$; is defined as

$$tanf_k(x) = \frac{sinf_k(x)}{cosf_k(x)} = \frac{e^{r_1x} - e^{r_2x}}{r_1e^{r_1x} - r_2e^{r_2x}}.$$

In this section, we will write $tanf_k(x)$ in terms of a power series of e^x.

Lemma 2.10 $r_2^n = F_{k,n-1} + F_{k,n}r_2$ and $r_1^n = F_{k,n-1} + F_{k,n}r_1$ for all $n \geq 1$.

Proof: We will prove by mathematical induction.

If $n = 1$ then $r_2 = F_{k,0} + F_{k,1}r_2 = r_2$.

If $n = 2$ then $r_2^2 = F_{k,1} + F_{k,2}r_2 = 1 + kr_2$. It is true for $n = 1$ and $n = 2$.

Suppose that the hypothesis is true for $n \geq 1$.

Namely, $r_2^n = F_{k,n-1} + F_{k,n}r_2$ is true. We will prove that the statement is true for r_2^{n+1}.

By the induction hypothesis, we have

$$\begin{aligned} r_2^{n+1} &= r_2(r_2^n) = r_2\left(F_{k,n-1} + F_{k,n}r_2\right) \\ &= F_{k,n-1}r_2 + F_{k,n}r_2^2 \\ &= F_{k,n-1}r_2 + F_{k,n}(1 + kr_2) \\ &= (F_{k,n-1} + F_{k,n}k)r_2 + F_{k,n}. \end{aligned}$$

Hence, $r_2^{n+1} = F_{k,n+1}r_2 + F_{k,n}$.

It is similar to prove $r_1^n = F_{k,n-1} + F_{k,n}r_1$ for all $n \geq 1$.

Theorem 2.11 The k -Fibonacci tangent can be written as

$$tanf_k(x) = -r_2 + (kr_2^2 + 2r_1)\sum_{n=0}^{\infty}\left[(-1)^n\left(F_{k,n-1} + F_{k,n}r_2\right)^2 \cdot e^{-(n+1)\sqrt{k^2+4}x}\right].$$

Proof: Observe that $tanf_k(x) = \dfrac{e^{r_1x} - e^{r_2x}}{r_1e^{r_1x} - r_2e^{r_2x}} = \dfrac{1 - e^{(r_2-r_1)x}}{r_1 - r_2e^{(r_2-r_1)x}}$.

We know that $r_1 - r_2 = \sqrt{k^2 + 4}$ and $r_1r_2 = -1$.

Hence,

$$tanf_k(x) = \frac{1 - e^{(r_2-r_1)x}}{r_1 - r_2e^{(r_2-r_1)x}} = r_2\Big(\frac{-1 + e^{-\sqrt{k^2+4}x}}{1 + r_2^2e^{-\sqrt{k^2+4}x}}\Big).$$

Since r_2 is the root of $r^2 - kr - 1 = 0$, we have $r_2^2 + 1 = kr_2 + 2$.

So,

$$\begin{aligned} tanf_k(x) &= r_2\Big(\frac{-1+e^{-\sqrt{k^2+4x}}}{1+r_2^2e^{-\sqrt{k^2+4x}}}\Big) \\ &= r_2\big[-1+(kr_2+2)e^{-\sqrt{k^2+4x}} - r_2^2(kr_2+2)e^{-2\sqrt{k^2+4x}} + r_2^4(kr_2+2)e^{-3\sqrt{k^2+4x}} - \cdots\big] \\ &= r_2\big[-1+\sum_{n=0}^{\infty}(-1)^n r_2^{2n}(kr_2+2)e^{-(n+1)\sqrt{k^2+4x}}\big] \\ &= r_2\big[-1+(kr_2+2)\sum_{n=0}^{\infty}(-1)^n r_2^{2n}e^{-(n+1)\sqrt{k^2+4x}}\big] \\ &= -r_2+(kr_2^2+2r_2)\sum_{n=0}^{\infty}(-1)^n r_2^{2n}e^{-(n+1)\sqrt{k^2+4x}}. \end{aligned}$$

Therefore,

$$tanf_k(x) = -r_2 + (kr_2^2+2r_2)\sum_{n=0}^{\infty}(-1)^n(F_{k,n-1}+F_{k,n}r_2)^2 e^{-(n+1)\sqrt{k^2+4x}}.$$

Definition 2.12 The k -Fibonacci cotangent, denoted by $cotf_k$; is defined as

$$cotf_k(x) = \frac{cosf_k(x)}{sinf_k(x)} = \frac{r_1e^{r_1x} - r_2e^{r_2x}}{e^{r_1x} - e^{r_2x}}.$$

Theorem 2.13 The k -Fibonacci cotangent can be written as $cotf_k(x) = -\frac{1}{r_2}\big(1+(kr_2+2)\big)\sum_{n=1}^{\infty} e^{-n\sqrt{k^2+4x}}$.

Proof: Since $cotf_k(x) = \frac{r_1e^{r_1x} - r_2e^{r_2x}}{e^{r_1x} - e^{r_2x}} = \frac{e^{r_1x} - \frac{r_2}{r_1}e^{r_2x}}{\frac{e^{r_1x}}{r_1} - \frac{e^{r_2x}}{r_1}} = \frac{1+r_2^2e^{-\sqrt{k^2+4x}}}{-r_2+r_2e^{-\sqrt{k^2+4x}}}$

$$\begin{aligned} &= \frac{1}{r_2}\Big[\frac{1+(kr_2+1)e^{-\sqrt{k^2+4x}}}{-1+e^{-\sqrt{k^2+4x}}}\Big] \\ &= \frac{1}{r_2}\big[-1-(kr_2+2)e^{-\sqrt{k^2+4x}} - (kr_2+2)e^{-2\sqrt{k^2+4x}} - \cdots\big]. \end{aligned}$$

Hence, $cotf_k(x) = -\frac{1}{r_2}\big(1+(kr_2+2)\big)\sum_{n=1}^{\infty} e^{-n\sqrt{k^2+4x}}$.

Before proving some elementary identities of k -Fibonometric functions we will give a definition of k -Fibonacci secant and cosecant.

Definition 2.14 The k -Fibonacci secant $(\sec f_k)$ and the k -Fibonacci cosecant $(\cos ecf_k)$ are defined as

$secf_k(x) = \dfrac{1}{cosf_k(x)}$ and $cosecf_k(x) = \dfrac{1}{sinf_k(x)}$.

3.1 Elementary Identities of k -Fibonometric Functions

Theorem 3.1 The following Identities are true.

(1) $cosf_k^{\,2}(x) - kcosf_k(x)sinf_k(x) - sinf_k^{\,2}(x) = e^{kx}$,

(2) $cotf_k^{\,2}(x) - kcotf_k(x) - 1 = e^{kx}cosecf_k^{\,2}(x)$,

(3) $1 - ktanf_k(x) - tanf_k^{\,2}(x) = e^{kx}secf_k^{\,2}(x)$,

(4) $sinf_k(x+y) = sinf_k(x)cosf_k(y) - ksinf_k(x)sinf_k y + cosf_k(x)sinf_k(y)$,

(5) $sinf_k(x-y) = sinf_k(x)cosf_k(-y) - ksinf_k(x)sinf_k(-y) + cosf_k(x)sinf_k(-y)$,

(6) $cosf_k(x+y) = cosf_k(x)cosf_k(y) + sinf_k(x)sinf_k(y)$,

(7) $cosf_k(x-y) = cosf_k(x)cosf_k(-y) + sinf_k(x)sinf_k(-y)$,

(8) $sinf_k(2x) = 2sinf_k(x)cosf_k(x) - ksinf_k^{\,2}(x)$,

(9) $cosf_k(2x) = cosf_k^{\,2}(x) + sinf_k^{\,2}(x)$.

Proof: The proof of (1) is as follows:

$$
\begin{aligned}
&cosf_k^{\,2}(x) - kcosf_k(x)sinf_k(x) - sinf_k^{\,2}(x)\\
&= \left(\frac{r_1e^{r_1x} - r_2e^{r_2x}}{r_1 - r_2}\right)^2 - k\left(\frac{r_1e^{r_1x} - r_2e^{r_2x}}{r_1 - r_2}\right)\left(\frac{e^{r_1x} - e^{r_2x}}{r_1 - r_2}\right) - \left(\frac{e^{r_1x} - e^{r_2x}}{r_1 - r_2}\right)^2\\
&= \frac{r_1^2e^{2r_1x} - 2r_1r_2e^{(r_1+r_2)x} + r_2^2e^{2r_2x} - kr_1e^{2r_1x} + kr_1e^{(r_1+r_2)x}}{(r_1 - r_2)^2}\\
&\quad + \frac{kr_2e^{(r_1+r_2)x} - kr_2e^{2r_2x} - e^{2r_1x} + 2e^{(r_1+r_2)x} - e^{2r_2x}}{(r_1 - r_2)^2}\\
&= \frac{(r_1^2 - kr_1 - 1)e^{2r_1x} + (kr_2 - 2r_1r_2 + kr_1 + 2)e^{(r_1+r_2)x} + (r_2^2 - kr_2 - 1)e^{2r_2x}}{(r_1 - r_2)^2}\\
&= \frac{(kr_2 - 2r_1r_2 + kr_1 + 2)e^{(r_1+r_2)x}}{(r_1 - r_2)^2} = \frac{k^2 + 4}{k^2 + 4}e^{(r_1+r_2)x} = e^{kx}.
\end{aligned}
$$

Therefore $cosf_k^{\,2}(x) - kcosf_k(x)sinf_k(x) - sinf_k^{\,2}(x) = e^{kx}$.

To prove (2). Observe that

$$cotf_k^2(x) - kcotf_k(x) - 1 = \frac{cosf_k^2(x)}{sinf_k^2(x)} - k\frac{cosf_k(x)}{sinf_k(x)} - 1$$

$$= \frac{cosf_k^2(x) - kcosf_k(x)sinf_k(x) - sinf_k^2(x)}{sinf_k^2(x)}$$

$$= \frac{e^{kx}}{sinf_k^2(x)} = e^{kx}cosecf_k^2(x).$$ This completes the proof of (2).

The proof of (3) is similar to (2).

Identity (4) can be seen from

$$sinf_k(x)cosf_k(y) - ksinf_k(x)sinf_k y + cosf_k(x)sinf_k(y)$$

$$= \frac{r_1e^{r_1(x+y)} - r_2e^{r_1(x+y)} - r_1e^{r_2(x+y)} + r_2e^{r_2(x+y)}}{(r_1 - r_2)^2} = \frac{(r_1 - r_2)e^{r_1(x+y)} - (r_1 - r_2)e^{r_2(x+y)}}{(r_1 - r_2)^2}$$

$$= \frac{e^{r_1(x+y)} - e^{r_2(x+y)}}{r_1 - r_2} = sinf_k(x + y).$$

The proofs of (5), (6)) and (7) are similar to (4).

The proofs of (8) and (9) are obvious from (4) and (6) respectively.

Results

In this paper, we have studied k -Fibonometric Functions and some identities of k -Fibonometric Functions.

Definition The k - Fibonacci sine, denoted by $sinf_k$; is defined as $sinf_k(x) = \frac{e^{r_1x} - e^{r_2x}}{r_1 - r_2}$

where $r_1 = \frac{k + \sqrt{k^2 + 4}}{2}$ and $r_2 = \frac{k - \sqrt{k^2 + 4}}{2}$.

Theorem The expansion of the k -Fibonacci sine is $sinf_k(x) = \sum_{n=0}^{\infty} \frac{F_{k,n}}{n!}x^n$ where $F_{k,n}$ is the n^{th} k - Fibonacci number.

Definition The k -Fibonacci cosine, denoted by $cosf_k$; is defined as $cosf_k(x) = \frac{r_1e^{r_1x} - r_2e^{r_2x}}{r_1 - r_2}$

where $r_1 = \dfrac{k+\sqrt{k^2+4}}{2}$ and $r_2 = \dfrac{k-\sqrt{k^2+4}}{2}$.

Theorem The expansion of the k -Fibonacci cosine is $cosf_k(x) = \sum_{n=0}^{\infty} \dfrac{F_{k,n+1}}{n!} x^n$.

Definition The k -Fibonacci tangent, denoted by $tanf_k$; is defined as

$$tanf_k(x) = \frac{sinf_k(x)}{cosf_k(x)} = \frac{e^{r_1 x} - e^{r_2 x}}{r_1 e^{r_1 x} - r_2 e^{r_2 x}}.$$

Theorem The k -Fibonacci tangent can be written as

$$tanf_k(x) = -r_2 + (kr_2^2 + 2r_1)\sum_{n=0}^{\infty}\left[(-1)^n \left(F_{k,n-1} + F_{k,n} r_2\right)^2 \cdot e^{-(n+1)\sqrt{k^2+4x}}\right].$$

Definition The k -Fibonacci cotangent, denoted by $cotf_k$; is defined as

$$cotf_k(x) = \frac{cosf_k(x)}{sinf_k(x)} = \frac{r_1 e^{r_1 x} - r_2 e^{r_2 x}}{e^{r_1 x} - e^{r_2 x}}.$$

Theorem The k -Fibonacci cotangent can be written as $cotf_k(x) = -\dfrac{1}{r_2}\left(1 + (kr_2 + 2)\right)\sum_{n=1}^{\infty} e^{-n\sqrt{k^2+4x}}$.

Elementary Identities of k -Fibonometric Functions

Theorem The following Identities are true.

(1) $cosf_k^2(x) - kcosf_k(x)sinf_k(x) - sinf_k^2(x) = e^{kx}$,

(2) $cotf_k^2(x) - kcotf_k(x) - 1 = e^{kx} cosecf_k^2(x)$,

(3) $1 - ktanf_k(x) - tanf_k^2(x) = e^{kx} secf_k^2(x)$,

(4) $sinf_k(x+y) = sinf_k(x)cosf_k(y) - ksinf_k(x)sinf_k y + cosf_k(x)sinf_k(y)$,

(5) $sinf_k(x-y) = sinf_k(x)cosf_k(-y) - ksinf_k(x)sinf_k(-y) + cosf_k(x)sinf_k(-y)$,

(6) $cosf_k(x+y) = cosf_k(x)cosf_k(y) + sinf_k(x)sinf_k(y)$,

(7) $cosf_k(x-y) = cosf_k(x)cosf_k(-y) + sinf_k(x)sinf_k(-y)$,

(8) $sinf_k(2x) = 2sinf_k(x)cosf_k(x) - ksinf_k^2(x)$,

(9) $cosf_k(2x) = cosf_k^2(x) + sinf_k^2(x)$.

Discussion

We have suggested how these may be similar to the hyperbolic functions and shown their series expansions in the role of function of e^x.

Conclusions

In this paper, we have defined the basic k-Fibonometric Functions and established some theorems and elementary identities for Fibonometry.

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